Logarithms Through Applications

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This curriculum writing project was inspired by the spirit of the NCTM Standards for School Mathematics in the hope that a similar emphasis on applications (connections), communication, and understanding could be incorporated into the standard college curriculum. It is designed to be part of a precalculus course, although it could also fit in Intermediate Algebra. There is probably enough material for two class periods.

Lesson 6.1 Exponentials, 6.2 Properties of Exponential Functions, and 6.3 Logarithms would precede this in the curriculum. I assume that students would have seen the compound interest formula and uses of the formula in modeling inflation. They should have also been exposed to exponential growth and decay.

In the lesson immediately preceding this one, the basic definition of logarithms would have been given (as the inverse of exponentials), and the properties of logarithms would have been deduced. Students should know the difference between common log and natural log, but realize that they have similar properties.

Earlier in the curriculum, students would have been exposed to variation (direct and inverse) and equations of lines through two given points. They should also have some knowledge of linear regression.

Students should know how to use their graphing calculator to plot functions, use the trace function, plot ordered pairs, and do linear regression. They should also be able to do computations involving exponential and logarithmic functions. In the precalculus course at Seton Hill, a computer algebra system (Maple V) was used instead of a graphing calculator.

My sources for applications include the following:

Example 1: A Sourcebook of Applications of School Mathematics (updated to 1990 information - "Critical Thinking" added)

Example 2: A Sourcebook of Applications (graphing added)

Example 3: A Sourcebook of Applications ("Try This" added)

Example 4: General knowledge

Exercise 6 and 7: Mathematics with Applications by Lial & Miller

Exercise 8 and 9: A Sourcebook of Applications.

Exercise 12 and 13: Idea "borrowed" from Algebra 2 by Cohen, et al. Actual exercises are mine.

Exercises 14 - 17: Equation obtained from an article by Fishman in Mathematics Teacher, November 1993. Exercises are mine.

Exercise 18: Mathematics with Applications.

6.4 Applications of Logarithms

Why The logarithms studied in the last section are used in mathematics and applications to:

1. solve exponential equations for variables appearing in the exponent

2. find a non-linear relationship between variables

3. model phenomena such as sound perception, earthquake intensity, flicker in a motion picture, PH level of foods, and city walking speed.

Remember:

$y = \log_b(x)$ is equivalent to $x = b^y$.

$\log_b(xy) = \log_b(x) + \log_b(y)$

$\log_b(x \div y) = \log_b(x) - \log_b(y)$

$\log_b(x^y) = y \log_b(x)$

Example 1

Ecology

In 1990, known world coal reserves were estimated at 8×10^{12} tons, but consumption was running at 5.8×10^9 tons per year. If consumption increases at a rate of 5% per year, how long will the coal last?

If consumption of coal was not increasing, we could easily calculate the time until known reserves would be exhausted. How? Using the fact that rate x time is a measure of total consumption, we see that

$$8 \times 10^{12} \text{ tons} = (5.8 \times 10^9 \text{ tons/year})(t \text{ years}).$$

So t = ______________ years. This is a very long time!

However, due to increasing industrialization in the developing countries, it is not reasonable to assume that coal consumption will remain constant. Instead, ecologists assume that it will increase at a constant rate each year (in our problem, it increases at 5% per year).

We saw in Lesson 6.2 that the compound interest formula could be used to model the effects of such things as inflation and the growth of consumption, if a constant percent increase, r, is assumed. In such cases, we saw that n (number of compounding periods per year) = ____. Hence, the formula: $A = P(1 + \frac{r}{n})^{nt}$

becomes: $A = P(1 + r)^t$.

Since P (the consumption in 1990) was 5.8×10^9, and r is assumed to be 5%, the consumption in year t (measured since 1990), A, is given by:

$$A(t) = 5.8(1.05)^t \text{ billion tons}$$

Graphics Calculator Use your graphics calculator to examine the graph of A(t) for t from 0 to 60. Use the trace function to answer the following:

1. What will A be in the year 2000 (t = ___)?

   __________ billion tons

2. What about 2025? ________________________ (units?)

3. When will consumption exceed 100 billion tons? ___

Logarithms The last question could have been answered with algebra and logarithms. For if A = 100, we have:

$$100 = 5.8(1.05)^t, \text{ so } \frac{100}{5.8} = 1.05^t.$$

By taking the common log of both sides, we obtain:

$$\log\left(\frac{100}{5.8}\right) = \underset{\text{Why?}}{\log(1.05)^t =} t\log(1.05)$$

Hence $t = \dfrac{\log(100/5.8)}{\log(1.05)} = 58.4$ years. Compare to #3!

So consumption of coal will reach 100 billion 58.4 years after 1990, during the year 2048.

More Logs

In order to determine when the known coal reserves (eight trillion tons in 1990) will run out, we need to be able to add the consumption in 1990, 1991, 1992, etc. until the year 1990 + t. In other words, we need to find:

$$C = A(0) + A(1) + A(2) + ... + A(t)$$

$$= P + P(1+r) + P(1+r)^2 + ... + P(1+r)^t$$

But a formula for any sum of the form:

$$S = a + aR + aR^2 + ... + aR^n$$ (**geometric series**)

is given by: $S = a \frac{R^{n+1} - 1}{R - 1}$.

Using R = 1+r and a = P, we get $C = P \frac{(1+r)^{t+1} - 1}{r}$.

In our case, $C = 5.8 \frac{1.05^{t+1} - 1}{.05}$.

To determine when the reserves will be depleted, we must substitute 8,000 for C (8,000 billion = 8 trillion) and then solve for 1.05^{t+1}:

$$1.05^{t+1} = 8000 \times .05 \div 5.8 + 1 = 69.9655$$

Now take the log of both sides:

$$\log(1.05^{t+1}) = \log(69.9655)$$

$$(t+1)\log(1.05) = 1.845$$

So t = 86 years.

Thus, if consumption of coal increases at a rate of 5% per year, the total known world reserves will be gone by 2076!

Critical Thinking

If the known world supply were doubled because of the discovery of a huge coal deposit, what would be the effect on the depletion date?

If the increase in consumption would be held to 4% per year, what would be the effect on t?

Which change has the greatest effect on t, an increase in supply or a decrease in the growth rate of consumption?

Example 2

Biology

An **allometry between x and y** exists if $y = b(x^k)$ for some b and k. Jill, a biologist believes that an allometry exists between baboon face length and skull length. She measures four baboons and obtains:

x face length (mm)	31.0	94.8	131.0	144.25
y skull length (mm)	78.5	108.9	114.7	122.0

What are the "best" estimates of b and k?

Solution >

In many cases, a logarithmic transformation converts a non-linear relationship into one that is linear. Try this with $y = b(x^k)$, by taking the log of both sides:

$$\log(y) = \log(b) + k \log(x) \quad \text{(why?)}$$

Thus if $y^* = \log(y)$, and $x^* = \log(x)$, we have:

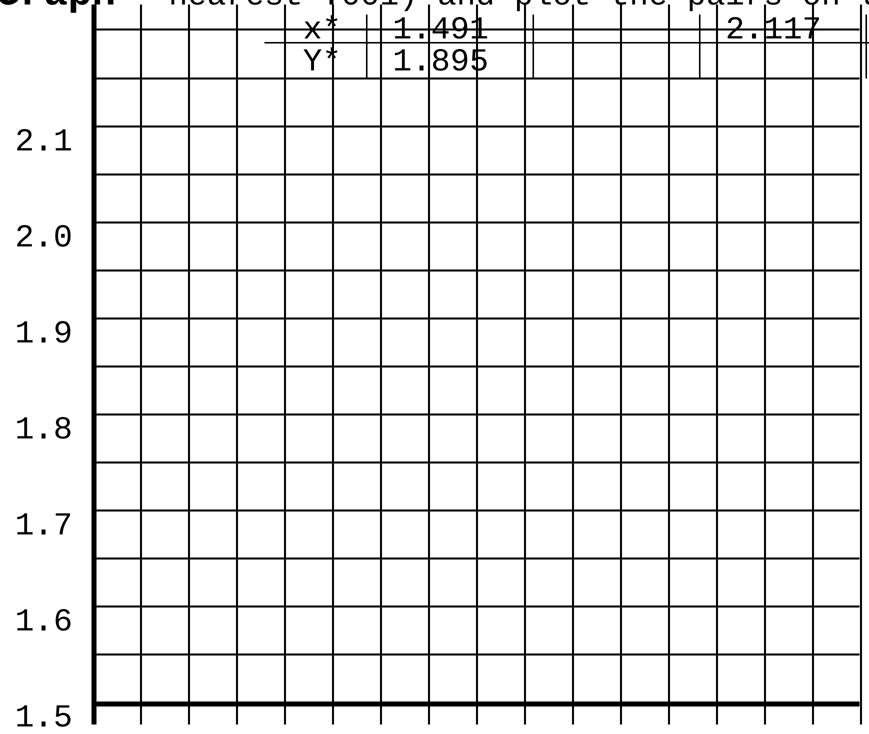
$$y^* = a_0 + a_1(x^*) \qquad \text{where } a_0 = \log(b) \text{ and } a_1 = k.$$

The relationship between x* and y* is linear!

Calculator and Graph

Fill-in the remainder of the following table (round to nearest .001) and plot the pairs on the provided grid:

x*	1.491		2.117	
Y*	1.895			2.086



1.4

1.4 1.5 1.6 1.7 1.8 1.9 2.0 2.1 2.2

Look at the graph that you've completed. Since the pairs seem to exhibit a linear relationship, it appears as though the biologist is right. But now, we need to find b and k. To do this, we'll use two different methods:

Method 1 - Graphical/Algebraic

1. Using the above graph, attempt to draw a straight line which "fits" the data as well as possible. As discussed earlier in the text, one possible strategy is to draw a line joining the "outside" points. Your line may differ slightly from others in your class.

2. Using two convenient points on your line, compute the slope. This is a_1.

3. Use $y^* = a_1x^* + a_0$ and any convenient point (x^*,y^*) from your line. Only a_0 is unknown, so you may find it via algebra.

Method 2 - Linear Regression

Use the statistical regression mode of your calculator to obtain the slope and y-intercept of a line which fits your data "best" (least squares criterion). You should obtain $a_0 = 1.4825$ and $a_1 = .2774$ (rounded).

Use your graphics calculator to plot the four (x^*,y^*) pairs and the linear regression line found. As in method 1, the line should closely fit the points.

The Allometry

In either method, we obtained values for a_0 and a_1, which gave a line fitting the data. In the following, we will use the values obtained by method 2:

Since $a_0 = \log(b)$, $b = 10^{a_0} = 30.3739$. $k = a_1 = .2774$ Thus, we have found the allometry for baboon face length and skull length:

$$y = 30.3739x^{.2774}.$$

Example 3

Perception

As you know, a movie is a sequence of still pictures which is shown fast enough to give the impression of motion to the viewer. If the frequency of the stills (pictures shown per second) is too small, the movie flickers; if the frequency is large enough, the brain perceives smooth motion.

According to research, the minimum frequency, f, at which the flicker first disappears is proportional to the logarithm of the light intensity, I, of the screen image reaching the viewer. Thus:

$f = K \log(I)$, K is the proportionality constant

We also know from previous work that I is inversely proportional to the square of the distance from the light source (the screen). Hence:

$I = k/d^2$, so $f = K \log(k/d^2) \underset{\text{why?}}{=} K[\log(k) - 2\log(d)]$

The Problem>

Suppose that Alicia notices a flicker from the film (she is sitting too close) and doubles her distance from the screen. What effect will this have on f?

The Solution>

Let f_1 be the minimum flicker-free frequency (a tongue twister!) in the original seat at distance d, and let f_2 be the frequency at distance 2d. Then:

$f_1 = K[\log(k) - 2\log(d)]$ and

$f_2 = K[\log(k) - 2\log(2d)] \underset{\text{Why?}}{=} K[\log(k) - 2\log 2 - 2\log(d)]$

So $f_1 - f_2 = K\log(4) = 0.602K$.

Hence, by doubling the distance from the screen, f is only reduced by 0.602K. This may not be enough to eliminate the flicker in the film!

Try This What if Alicia triples her distance from the screen. What is the effect on f?

Critical Thinking Suppose Karl cannot change seats (the movie theater is full), but he perceives a flicker. How can he solve the problem? Hint: consider f = K log(I).

Example 4

Chemistry The pH factor determines whether a substance is alkaline, neutral, or acidic depending if pH is greater than, equal to, or less than 7 respectively. If H^+ is the hydrogen ion concentration in moles, pH is determined by: $pH = -\log(H^+)$.

1. Tomato juice has $H^+ = 6.3 \times 10^{-5}$ moles. What is the pH value?

   $pH = -\log(6.3 \times 10^{-5}) = 4.2$

   So tomato juice is acidic.

2. Milk of magnesia has $H^+ = 3.2 \times 10^{-11}$.

   pH is ______ and milk of magnesia is _____________

Critical Thinking What is the H^+ value for a neutral substance?

Exercises & Problems

Communicate

1. Explain why the natural logarithm could have been used in example 1 and 2, when we took the logarithm of both sides of an exponential equation.

2. In example 3, common logs were used to model the frequency at which movie flicker disappears. Could natural logs be used throughout? Explain!

3. Give the product, quotient, and power rules for logarithms in your own words. E.g. "The log of a product is ..."

Practice and Apply

4. If $y = a(b^x)$, what transformation would you use to create a linear relationship between x* and y*?

Business A famous quote by Ben Franklin says "A penny saved is a penny earned." But suppose that you deposit the penny in a savings account paying 5% interest, compounded monthly.

5. How long would it take until the amount is $1,000? Until it is $1 million? Assume that the bank does not round interest accumulations until the money is finally withdrawn.

Business **The Rule of 72** states that it will take approximately 72 ÷ r years for a deposit to double, if interest is compounded annually at r%.

6. Using the compound interest formula and logarithms, find how long it would take until your $1 deposit doubles if the rates are 4%, 6%, and 8% (round to the nearest year).

7. Compare the answers obtained in #6 with those given by the Rule of 72.

Biology Michaela believes that there is an allometry between the height, x, and the weight, y, of humans. A study reveals the average weights for boys and girls of heights 60 and 70 inches are:

	Weight	
Height	Boys	Girls
60	104	94
70	157	143

8. Find the allometry relationship for boys (use log transformations and the point-slope formula).

9. Find the allometry relationship for girls.

Astronomy Let d be the average distance of a planet from the sun, measured in **astronomical units (A.U.)**. One A.U. is the average distance of Earth from the sun, approximately 93 million miles. Let T be the **period** of the planet, defined to be the length of time in years required to make one orbit of the sun. The following table gives d and T for the nine planets:

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
d	0.387	0.723	1.000	1.523	5.203	9.541	19.190	30.086	39.507
T	0.241	0.615	1.000	1.881	11.861	29.457	84.008	164.784	248.350

10. Find a relationship between log(T) and log(d).

11. Use the result of #10 to find the relationship that Kepler found in 1620, giving T as a function of d.

Sociology It has been observed that walking speed, s, measured in miles per hour is related to the population of a city, by:

$$s = 1.38 \log(p) + .03$$

12. Find the walking speed in Pittsburgh (population 370,000), New York (7.3 million), and Greensburg, PA (16,000).

13. What is the population of a town, if the walking speed is 5.55 mph?

Ecology An analysis of the yearly consumption, A, of natural gas in the United States between 1900 and 1970 showed that:

$A(t) = 0.3(1.063)^t$ trillion cubic feet (t.c.f.), for $0 \le t \le 70$

14. Use your graphics calculator to graph A(t). Use the trace function to determine when A reached 3 t.c.f.

15. Use logarithms to determine when A reached 20 t.c.f.

16. Find the total consumption between 1900 and 1970.

17. Data shows that United States consumption of natural gas ceased its exponential growth after 1970. In fact, in 1990 A was 18.55 t.c.f. What would the consumption have been in 1990, if the same growth had continued after 1970?

Archeology As discussed in Lesson 6.2, carbon-14 (C-14) decay is modeled by an exponential function:

$y = y_0 e^{-.000123t}$, where y is the amount of C-14 remaining t years after a living organism's death.

18. A round table hanging in Winchester Castle (Great Britain) was supposed to be the famous table used by King Arthur of the 5th century. Chemical analysis in the 1950's showed that the table had 91% of the amount of C-14 present in living wood ($.91y_0$). Find the approximate age of the table. Was it King Arthur's round table?

Text Answers:

Page 1 t = 1,379 years

Page 2 n (number of compounding periods per year) = 1
1. (t = 10) 9.5 billion tons
2. 32.2 billion tons
3. 2048

Why?: power rule for logarithms
Compare to #3 -- 58.4 gives 2048, which is the same year as found in #3

Page 3 Critical Thinking
1. If deposits doubled, C = 16,000 which gives t=100, or 14 more years of consumption
2. If r is 4% (and C = 8,000), t=107, so 21 more years.
3. The decrease in growth rate has the greatest effect -- C changed by 100%, yielding a change of 14 years in t, but r changed by only 20%, and yielded a 21 year change

Page 4 Why?: $\log(bx^t) = \log(b) + \log(x^t)$ by product rule
$= \log(b) + t\log(x)$ by power rule

Table:

1.491	1.977	2.117	2.159
1.895	2.037	2.060	2.086

Page 6 Why?: $K\log(k/d^2) = K[\log(k) - \log(d^2)]$ quotient rule
$= K[\log(k) - 2\log(d)]$ power rule
Why?: $-2\log(2d) = -2[\log(2) + \log(d)]$ product rule

Page 7 Try: $\log(3d) = \log(3) + \log(d)$, so $f_1 - f_2 = K(2\log(3))$
$= K\log(9) = .954K$
Critical: reduce I by wearing sunglasses or squinting
Milk of magnesia: pH is 10.5 and it is alkaline.
Critical: pH = 7, so $-7 = \log(H^+)$, so $H^+ = 10^{-7}$ moles.

Answers to Selected Exercises

1. Answers vary, but the main idea is that the properties of logs used apply for logs of any base.

2. Natural log could have been used since $\ln(I) = \log(I)/\log(e)$ and so $K\ln(I) = (1/\log(e))K\log(I)$. Just use a different constant of variation.

3. The log of a product is the sum of the logs of the factors.
The log of a quotient is the difference of the logs.
The log of an exponential is the exponent multiplied by the log of the base.

4. Since log(y) = log(a) + x(log(b), use y* = log(y) and x* = x Then $y^* = a_0 + a_1x^*$ is linear, with a_0 = log(a) and a_1 = log(b).

5. $A = .01(1 + .05/12)^n$.
   a. If A=1000, log(1000/.01) = n log(1.00417)
      n = 2766.6 months = 230.6 years.
   b. If A=1,000,000 a similar calculation gives 4426.6 months or 368.9 years.

   NOTE: The assumption is required -- otherwise the bank would constantly pay 0.00 interest on the original penny, and hence the money would never grow.

6. $2 = 1(1 + r)^n$. So log(2) = n log(1 + r),
   and n = log(2) ÷ log(1 +r).
   If r = 4%, n = 17.7 years = 18 years (rounded).
   If r = 6%, n = 11.9 years = 12 years.
   if r = 8%, n = 9.01 years = 9 years.

7. Rule of 72 gives 72/4 = 18, 72/6 = 12, and 72/8 = 9 years.

10. log(T) = 1.5 log(d)

11. $\log(T) = \log(d^{1.5})$, so $T = d^{1.5}$ or $T^2 = d^3$.

12. Pittsburgh: 1.38 log(370,000) + .03 = 7.71 mph.
    Similarly, NYC: 9.50 and Greensburg: 5.83 mph.

13. 5.55 = 1.38 log(p) + .03, so log(p) = 4, so $p = 10^4$.

14. Approximately 1937 (t=37.9)

15. $20 = .3(1.063)^t$, so log(20/.3) = t log(1.063).
    Thus t = 68.7, so 1968.

18. $.91y_0 = y_0e^{-.000123t}$. Thus $\ln(.91) = \ln(e^{-.000123t})$.
    So -.000123t = ln(.91) and t = 767 years old. It cannot be King Arthur's Table, since the tree was cut down around the year 1200.