Mathematics Competition

Indiana University of Pennsylvania

1997

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.

2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.

3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.

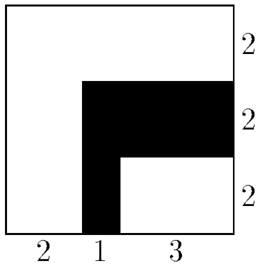
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.

5. Use of pencil, eraser, and scratch paper only are permitted.

6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. If the lengths, in units, of the line segments are as indicated, then the area, in square units, of the shaded region is:



(A) 6
(B) 10
(C) 16
(D) 24
(E) 32

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2. The acceleration a that results when the force F is applied to a body of mass m can be calculated from the formula $F = ma$. If $m = 1200$ and $F = 90{,}000$, then a is equal to:

(A) 75 (B) 750 (C) 7500 (D) 1,080,000 (E) 108,000,000

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3. There is a 10% chance that the buyer of a new car will experience a problem with the car in the first three months of operation. If a company purchases two cars for its staff, the chance that exactly one of these cars will have a problem in the first three months of operation is:

(A) 7% (B) 10% (C) 18% (D) 20% (E) none of these

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4. Salazar only ran 50 miles in one week, while Rodgers ran twice as many miles as Salazar. Frank ran three times as many miles as Salazar and Rodgers put together. If Mike were to run 10% of the miles Salazar and Frank ran combined, then Mike would run:

(A) 15 miles (B) 50 miles (C) 100 miles (D) 450 miles
(E) none of these

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5. The solution set of the equation

$$\log_4(x+6) - \log_4 10 = \log_4(x-1) - \log_4 2$$

is:

(A) $\{4/3\}$ (B) $\{27/19\}$ (C) $\{3\}$ (D) $\{11/4\}$ (E) none of these

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6. If $x^2 \cdot 1995^2 \cdot 1996^2 \cdot 1997^2 = 3990^2 \cdot 3992^2 \cdot 3994^2$ and $x > 0$, then x is equal to:

(A) $\sqrt{6}$ (B) 16 (C) 8 (D) 64 (E) none of these

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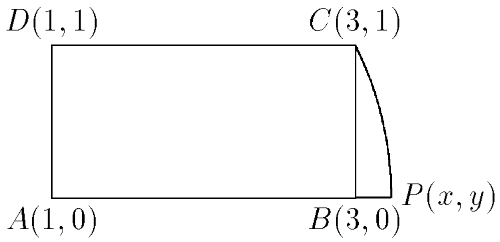
7. In the figure below, $ABCD$ is a rectangle with $AB = 2AD$. If P lies on the line going through A and B and CP is an arc of a circle centered at A, then the coordinates of the point P are:



(A) $(4,0)$
(B) $(\sqrt{5},0)$
(C) $(1+\sqrt{5},0)$
(D) $(3+\sqrt{5},0)$
(E) none of these

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8. The fourth power of $\sqrt{1+\sqrt{2}}$ is:

(A) $5+2\sqrt{2}$ (B) $3+\sqrt{2}$ (C) 5 (D) $3+2\sqrt{2}$ (E) none of these

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9. For every real number t, the expression $\cos^4 t - \sin^4 t$ is equal to:

(A) 0 (B) $\cos^2 2t$ (C) $\cos 2t$ (D) 1 (E) none of these

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10. The domain of the function defined by $f(x) = \sqrt{\sqrt{x-2}-1}$ is:

(A) $(-\infty,\infty)$ (B) $[0,\infty)$ (C) $[1,\infty)$ (D) $[2,\infty)$ (E) $[3,\infty)$

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11. The solution set of the equation

$$\frac{x}{x-3} = \frac{3}{x-3} + 9$$

is:

(A) $\{\ \}$ (B) $\{3\}$ (C) $\{9\}$ (D) $\{-3\}$ (E) $\{3,9\}$

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12. If the diameter of a circle is increased by 3π units, then the amount by which the circumference is increased is:

(A) 3π units (B) $3\pi^2$ units (C) $6\pi^2$ units (D) $\frac{3}{2}\pi^2$ units
(E) none of these

---

13. A store was having a hard time selling its frying pans so it marked the pans down 20%. The pans still would not sell and were marked down an additional 20% from the sale price. The pans now sold for \$16 each. The original price of a frying pan was:

(A) \$22.40 (B) \$23.04 (C) \$25.00 (D) \$32.00 (E) none of these

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14. Carol's average bowling score for 3 games was 138, and her highest score for the 3 games was 24 points higher than her average score. The information that **cannot** be determined is:

(A) Carol's highest score
(B) Carol's lowest score
(C) the sum of Carol's lowest two scores
(D) the sum of Carol's scores for the 3 games
(E) the difference between her highest score and her average score for the 3 games

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15. The solution set of the equation

$$\sqrt{3x+4}+x=0$$

is:

(A) $\{-1\}$ (B) $\{4\}$ (C) $\{-1,4\}$ (D) $\{\ \}$ (E) none of these

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16. The solution set of the equation

$$\log_{1/2} 16 = -x^2$$

is:

(A) $\{1/8\}$ (B) $\{2\}$ (C) $\{4\}$ (D) $\{4,-4\}$ (E) $\{2,-2\}$

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17. If the volume of a sphere is 16π cubic inches, then the volume in cubic inches of a cube all of whose faces are tangent to the sphere is:

(A) 64 (B) 96 (C) $16\sqrt[3]{\pi}$ (D) 512 (E) none of these

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18. If the sum of two numbers is 28 and their product is 7, then the sum of their reciprocals is:

(A) $\frac{11}{28}$ (B) $\frac{7}{28}$ (C) 4 (D) 6.9 (E) none of these

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19. Thirty people will attend the church's annual pancake and sausage breakfast. Four dozen sausages are ordered. All of the sausage is to be served and each person is to receive the same amount of sausage. The fewest cuts necessary to achieve this result is:

(A) 90 (B) 30 (C) 24 (D) 18 (E) none of these

---

20. George and Martha are going to shovel snow off their driveway. George figures that it would take him three times as long to do it alone as it will take them together. If they both work steadily, the portion of the driveway that Martha will clear is:

(A) 3/4 (B) 2/3 (C) 1/2 (D) 1/3 (E) impossible to determine

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21. The solution set of the equation

$$(\ln x)^2 + 2\ln x + 1 = 0$$

is:

(A) $\{\ \}$ (B) $\{1\}$ (C) $\{e\}$ (D) $\{1/e\}$ (E) $\{0\}$

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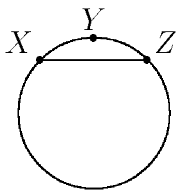
22. A circle has radius 2 and arc XYZ on the circle has length π. The length of the line segment $\overline{XZ}$ is:

(A) $\sqrt{2}$
(B) 2
(C) $2\sqrt{2}$
(D) $\pi/2$
(E) none of these



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23. The solution set of the equation

$$\left(\frac{x}{x-1}\right)^2 - 4\left(\frac{x}{x-1}\right) + 3 = 0$$

is:

(A) $\{\ \}$ (B) $\{1\}$ (C) $\{3/2\}$ (D) $\{2/3\}$ (E) $\{0, 3/2\}$

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24. If $f(x) = x^3$ and $h \neq 0$, then

$$\frac{f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)}{h^3}$$

is equal to:

(A) 0 (B) $a^3 + 3a^2h + 3ah^2 + h^3$ (C) $a^2 + 2ah + h^2$ (D) 6 (E) 3

---

25. Given the inequalities $\frac{d}{r_1} + \frac{d}{r_2} \geq t$, $r_1 r_2 < 0$, and $r_1 + r_2 > 0$, it follows that:

(A) $d \leq \frac{t}{r_1 + r_2}$ (B) $d \geq \frac{t}{r_1 + r_2}$ (C) $d \leq \frac{t r_1 r_2}{r_1 + r_2}$ (D) $d \geq \frac{t r_1 r_2}{r_1 + r_2}$
(E) none of these

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26. The set of solutions to the equation $\sin 2x = \cos x$ that lie in the interval $[0, 2\pi]$ is:

(A) $\{\tan^{-1}(.5)\}$ (B) $\{\tan^{-1}(.5), \pi + \tan^{-1}(.5)\}$ (C) $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$
(D) $\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}\right\}$ (E) none of these

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27. If John gets 97 on his next math test, his average will be 90. If he gets 73, his average will be 87. The number of tests that John has already taken is:

(A) 6 (B) 7 (C) 8 (D) 9 (E) none of these

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28. The solution set to the equation

$$\sqrt{2x+1} - \sqrt{x} = 5$$

is:

(A) $\{24\}$ (B) $\{4\}$ (C) $\{144\}$ (D) $\{4, 144\}$ (E) none of these

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29. The area, in square units, of an equilateral triangle whose side is 1 unit longer than its altitude is:

(A) $7 + 2\sqrt{3}$ (B) $12 + \sqrt{7}$ (C) 3 (D) $12 + \sqrt{3}$ (E) none of these

---

30. The exact value of $\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}$ is:

(A) 0 (B) 6 (C) $\sqrt{12}$ (D) -2 (E) 3

---

31. The value of $(\log_2 3)(\log_3 4)(\log_4 5)$ is equal to:

(A) 5 (B) $\log_2 5$ (C) 4 (D) $\ln 4$ (E) 3^4

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32. The fraction equivalent to .52141414... (where the 14's repeat forever) is:

(A) $\frac{2581}{4950}$ (B) $\frac{1738}{3333}$ (C) $\frac{869}{1650}$ (D) $\frac{439}{825}$ (E) none of these

---

33. The number of real solutions of the equation

$$x^{3/2} - 2x + 3x^{1/2} - 6 = 0$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 6

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34. The exact value of

$$\cos\frac{\pi}{20} + \cos\frac{2\pi}{20} + \cos\frac{3\pi}{20} + \cdots + \cos\frac{40\pi}{20}$$

is:

(A) 0 (B) 1 (C) -1 (D) π (E) e

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35. If p, q, and r are chosen so that the equation

$$x^4 + 4x^3 - 2x^2 - 12x + 9 = (px^2 + qx + r)^2$$

holds true for all real numbers x, then the product pqr is:

(A) 6 (B) -6 (C) 12 (D) -12 (E) not uniquely determined

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36. If the line $3x + 4y = 14$ is a tangent line to the graph of $x^2 + y^2 - 10x - 12y + b = 0$, then b must equal:

(A) 24 (B) 28 (C) 32 (D) 38 (E) none of these

---

37. Three boys and four girls line up randomly for a photograph (all in one row). The probability that the three boys are all next to each other is:

(A) $\frac{1}{35}$ (B) $\frac{1}{70}$ (C) $\frac{2}{7}$ (D) $\frac{1}{7}$ (E) none of these

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38. The solution, in interval notation, of the inequality

$$\frac{x(3-x)(x^2-4)}{x+5} \geq 0$$

is:

(A) $(-\infty, \infty)$ (B) $(-\infty, -5] \cup [-2, 0] \cup [2, 3]$
(C) $(-5, -2] \cup [0, 2] \cup [3, \infty)$ (D) $[-5, -2] \cup [0, 2] \cup [3, \infty)$
(E) $(-\infty, -5) \cup [-2, 0] \cup [2, 3]$

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39. The sides of a triangle are 30, 70, and 80 units. If an altitude is dropped upon the side of length 80, then the length of the larger segment cut off on this side is:

(A) 61 units (B) 62 units (C) 63 units (D) 64 units (E) 65 units

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40. The solution set of the equation

$$\sqrt{x+10} + \sqrt[4]{x+10} = 2$$

is:

(A) $\{6\}$ (B) $\{7\}$ (C) $\{8\}$ (D) $\{10\}$ (E) none of these

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41. Suppose $\log_y x = \log_x y$ where x and y are positive real numbers with $x \neq y$. It follows that:

(A) $xy = 1$ (B) $x + y = 1$ (C) $x + y = 2$ (D) $x^y = y^x$
(E) none of these

---

42. Let $S = \{3k \mid k \text{ is a positive integer}\}$. If the sum of the smallest $3n$ integers in S is 75,150 more than the sum of the smallest $2n$ integers in S, then n is:

(A) 96 (B) 99 (C) 102 (D) 105 (E) none of these

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43. Two investors, A and B, have \$10,000 each. They both split their money among three accounts with annual yields of 8%, 6%, and 5% respectively, but investor A only puts half as much as B in the 8% and 6% accounts. If investor A earns \$580 in interest annually while B earns \$660, then the amount that B must have invested at 8% is:

(A) \$2000 (B) \$4000 (C) \$5000 (D) \$2400
(E) not uniquely determined

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44. Suppose that in a triangle ABC, the sides $AB = 13$, $BC = 14$, and $CA = 14$. If $\overline{DC}$ and $\overline{EA}$ are altitudes from C and A respectively, then DE is equal to:

(A) 6.5 (B) 7 (C) 12.4 (D) 6 (E) none of these

---

45. In a rectangle $ACED$, side $\overline{EC}$ is extended through C to a point B so that $\overline{BD}$ bisects $\angle ABC$. If $AC = 4$ and $BC = 3$, then the length of $\overline{BD}$ is:

(A) $\sqrt{241}$ (B) $4\sqrt{5}$ (C) $\sqrt{65}$ (D) $2\sqrt{13}$ (E) none of these

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46. The largest real root of the polynomial $p(x) = 2x^4 + 9x^3 - 2x$ is:

(A) $1/2$ (B) 0 (C) $-1/2$ (D) $2 - \sqrt{6}$ (E) none of these

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47. Consider the point $P(5, 3)$, the curve $x^2 + y^2 + 8x - 10y + 5 = 0$, and the point Q where Q is the point of intersection of a tangent line to the curve through P and the curve itself. The length of the line segment $\overline{PQ}$ is:

(A) $\sqrt{85}$ (B) $6 + \sqrt{5}$ (C) 7 (D) 11 (E) none of these

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48. The number of digits in the decimal expansion of $8^6 \cdot 5^{12}$ is:

(A) 13 (B) 14 (C) 18 (D) 100 (E) none of these

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49. Let

$$\begin{aligned} a &= w + x + y + z, \\ b &= wx + wy + wz + xy + xz + yz, \\ c &= wxy + wxz + wyz + xyz, \\ d &= wxyz. \end{aligned}$$

Then $w^2x^2 + w^2y^2 + w^2z^2 + x^2y^2 + x^2z^2 + y^2z^2$ is equal to:

(A) b^2 (B) $b^2 - 2ac$ (C) $b^2 - 2ac - 6d$ (D) $b^2 - 2ac + 2d$
(E) none of these

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50. Two circles of radius 1 are externally tangent. A third circle of radius 3 is drawn so that both of the original circles are internally tangent to it. Finally a fourth circle is drawn internally tangent to the large circle and externally tangent to both of the original circles. The radius of this last circle is:

(A) 2 (B) $\dfrac{3\sqrt{3} - 1}{2}$ (C) $\dfrac{15 - 6\sqrt{3}}{13}$ (D) $\dfrac{15 + 6\sqrt{3}}{13}$

(E) none of these

Answer Key

1. B
2. A
3. C
4. B
5. D
6. C
7. C
8. D
9. C
10. E
11. A
12. B
13. C
14. B
15. A
16. E
17. B
18. C
19. C
20. B
21. D
22. C
23. C
24. D
25. C
26. D
27. B
28. C
29. E
30. E
31. B
32. A
33. B
34. A
35. E
36. E
37. D
38. E
39. E
40. E
41. A
42. E
43. E
44. A
45. B
46. E
47. C
48. B
49. D
50. D