Mathematics Competition

Indiana University of Pennsylvania

1998

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.

2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.

3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.

4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.

5. Use of pencil, eraser, and scratch paper only are permitted.

6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. The greatest odd factor of 7,344 is:

(A) 27 (B) 51 (C) 153 (D) 207 (E) none of these

---

2. The value of $\sqrt{\sqrt{\left(\sqrt{\left(\sqrt{\left(\sqrt{2^2}\right)^2}\right)}^4\right)}^2}$ is:

(A) 6.4 (B) 2 (C) $\sqrt{2}$ (D) $\sqrt[16]{2}$ (E) none of these

---

3. If two concentric circles have diameters of measure 2 units and 4 units respectively, then the area in square units between the inner and outer circles is:

(A) π (B) 3π (C) 4π (D) 12π (E) none of these

---

4. Let $f(x) = x^3 - 10x^2 + 25x + 6$. The number of distinct values a such that $f(a) = 6$ is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) none of these

---

5. If $x = (\log_{49} 7)^{\log_7 49}$, then x is:

(A) $1/4$ (B) $\sqrt{2}$ (C) $-1/4$ (D) 1 (E) none of these

---

6. Of the following equations, the one in which y is neither directly nor inversely proportional to x is:

(A) $2x + 3y = 0$ (B) $5xy = 7$ (C) $\dfrac{x}{y} = 2\sqrt{3}$ (D) $3x + 4y = 7$

(E) $y = \dfrac{7\sqrt{2}}{x}$

---

7. If a standard deck of 52 cards is well shuffled, the probability that the top two cards do not form a pair is:

(A) $\dfrac{51}{52}$ (B) $\dfrac{12}{13}$ (C) $\dfrac{47}{51}$ (D) $\dfrac{16}{17}$ (E) none of these

---

8. If the ratio of $a + x$ to $b - x$ is c where $c > 1$, then x is equal to:

(A) $\dfrac{a+b}{c-1}$ (B) $\dfrac{a-bc}{c}$ (C) $\dfrac{bc-a}{1+c}$ (D) $\dfrac{bc+a}{c-1}$ (E) none of these

---

9. The radian measure of the direction angle for the vector $\langle -1, 1\rangle$ is:

(A) $\frac{3\pi}{4}$ (B) -1 (C) undefined (D) $90°$ (E) $\frac{2\pi}{3}$

---

10. The value of $(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{63} 64)$ is:

(A) 6 (B) $\log_{64} 65$ (C) 1 (D) $\log_2(64!/2)$ (E) e^{63}

---

11. The expression $3+\sqrt{3}+\frac{1}{3+\sqrt{3}}+\frac{1}{\sqrt{3}-3}$ is equal to:

(A) $3+\frac{2\sqrt{3}}{3}$ (B) $4+\sqrt{3}$ (C) $3+\frac{4\sqrt{3}}{3}$ (D) $2+\frac{\sqrt{3}}{3+\sqrt{3}}$

(E) none of these

---

12. The measure of an interior angle of a regular hexagon is:

(A) $720°$ (B) $60°$ (C) $30°$ (D) $120°$ (E) none of these

---

13. Solving the equation

$$3x^2y^2+2xy-1=0$$

for y in terms of x, one gets:

(A) $y=\frac{-2x\pm\sqrt{11x^2}}{6x^2}$ (B) $y=\frac{1}{x}$ or $y=\frac{-1}{3x}$ (C) $y=\frac{-2x\pm\sqrt{2x+12x^2}}{3x^2}$

(D) $y=\frac{1}{3x}$ or $y=\frac{-1}{x}$ (E) none of these

---

14. If $f(x)=3^x$, then $f(x+2)-f(x+1)$ is equal to:

(A) $f(x)$ (B) $3f(x)$ (C) $6f(x)$ (D) $9f(x)$ (E) none of these

---

15. Suppose the ratio of $4x+3$ to $2y-1$ is constant. If $y=2$ when $x=1$, then when $y=5$ the value of x is:

(A) 6 (B) $9/2$ (C) 4 (D) $23/7$ (E) none of these

---

16. The solution set of the equation

$$\log_3 x + \log_9 x + \log_{27} x = 11$$

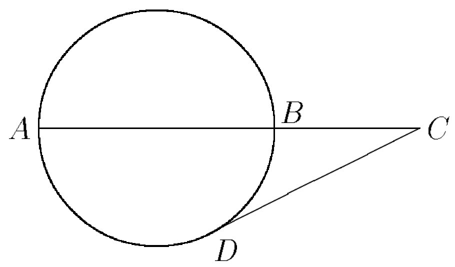
is:

(A) $\{\ \}$ (B) $\{2, 4, 8\}$ (C) $\{3, 9, 27\}$ (D) $\{729\}$ (E) none of these

---

17. In the accompanying figure, $\overline{AB}$ is a diameter of the circle, and $\overline{DC}$ is tangent to the circle. If $BC = 5$ and $DC = 8$, then the diameter of the circle to the nearest tenth of a unit is:

(A) 5.8
(B) 7.8
(C) 5.0
(D) 7.0
(E) none of these



---

18. If the arithmetic mean of $x^{1/2}$ and $x^{1/4}$ is 6, then x is equal to:

(A) 64 (B) 81 (C) 144 (D) 256 (E) none of these

---

19. The number of four-digit numbers that are divisible by 4 that can be formed from the digits 1, 2, 3, 4, and 5 with repetitions allowed is:

(A) 5^2 (B) 5^3 (C) 5^4 (D) 5^5 (E) none of these

---

20. Given that the sum of two of the roots of $4x^3 + 16x^2 - 9x - 36 = 0$ is zero, the solution set for this equation is:

(A) $\{-4, \frac{5}{2}, 4\}$ (B) $\{-\frac{2}{3}, \frac{2}{3}, 4\}$ (C) $\{-\frac{1}{2}, \frac{1}{2}, 3\}$ (D) $\{-3, \frac{1}{2}, 3\}$
(E) $\{-4, -\frac{3}{2}, \frac{3}{2}\}$

---

21. If $a^x b^x = c^y$, then y is:

(A) $\dfrac{x \log_{10} c}{\log_{10}(ab)}$ (B) $\dfrac{\log_{10}(a^x b^x)}{c}$ (C) $[x \log_{10}(ab)]^{1/c}$
(D) $x(\log_{10} a + \log_{10} b) - \log_{10} c$ (E) $\dfrac{x(\log_{10} a + \log_{10} b)}{\log_{10} c}$

---

22. A triangle has vertices at the points $(2, 6)$, $(6, 0)$, and $(8, 10)$. The area of this triangle in square units is:

(A) 12 (B) 17 (C) 24 (D) 26 (E) none of these

---

23. The set of ordered pairs of real numbers (x, y) that satisfy the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{x+y}$$

is:

(A) $\{\ \}$ (B) $\mathbb{R}^2$ (C) $\{(0,0)\}$ (D) $\{(x, y) \mid x \neq 0, y \neq 0, y \neq -x\}$
(E) none of these

---

24. Consider an isosceles triangle with an inscribed circle of radius 1. If one of the equal angles is α, then the length of each of the equal sides is:

(A) $\sec\alpha$ (B) $2\cot(\alpha/2)$ (C) $\tan\alpha + \frac{1}{2}\cot\alpha$ (D) $\dfrac{\cot(\alpha/2)}{\cos\alpha}$

(E) none of these

---

25. A chemist has 8% and 15% acid solutions and needs to mix them to obtain 2 liters of a 10% acid solution. The number of liters of 8% solution in the mix must be:

(A) 1 (B) 2 (C) $15(2 - x)$ (D) $10/7$ (E) .08

---

26. Functions f and g are defined by $f(x) = \sqrt{10 - x}$ and $g(x) = \sqrt{x - 4}$. The domain of the composite function $f \circ g$ is:

(A) $[4, \infty)$ (B) $(-\infty, 10]$ (C) $[4, 10]$ (D) $(-\infty, 104]$
(E) none of these

---

27. Given a circle of radius 2 m, the difference between the perimeters of a circumscribed square and an inscribed square measured in meters is:

(A) $8(2 - \sqrt{2})$ (B) 8 (C) $16 - 8\sqrt{3}$ (D) 16π (E) none of these

---

28. The solution set, in interval notation, of the inequality

$$x^3 - 4x^2 - 2x + 8 \geq 0$$

is:

(A) $[0, 4]$ (B) $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, 4]$ (C) $[-\sqrt{2}, \sqrt{2}] \cup [4, \infty)$
(D) $[-\sqrt{2}, 4]$ (E) $(-\infty, \infty)$

---

29. An operation $\#$ is defined by $x \# y = x + y - xy$ on the set of rational numbers and has zero as its identity. The inverse of $4/7$ under the operation $\#$ is:

(A) $1/8$ (B) $-4/7$ (C) $7/4$ (D) $-4/3$ (E) none of these

---

30. The number of ordered triples of real numbers that are solutions to the system

$$\begin{aligned} 2z &= 4 \\ x - 2y + z &= -1 \\ -2x + 4y - 2z &= 2 \end{aligned}$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) infinite

---

31. Suppose $\log_a 2 + \frac{1}{2}\log_a b - \log_a(b-1) = c$ where $b, c > 0$ and $a > 1$. It then must be true that:

(A) $0 < b < 3$ (B) $0 < b < 4$ (C) $1 < b < 4$ (D) $1 < b < 6$
(E) $2 < b < 6$

---

32. The distance from the point $(3, -2)$ to the line $3x - 4y - 6 = 0$ is:

(A) $7/10$ (B) $10/7$ (C) $11/5$ (D) $5/11$ (E) none of these

---

33. If x and y are nonnegative integers such that $x \leq \frac{3}{2}y + 1$ and $y < \frac{1}{3}x$, then $x - y$ is equal to:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

---

34. A regular pentagon has area 10,000 square units. The length of a side of this pentagon is:

(A) $40\sqrt{5\tan 54^\circ}$ (B) $20\sqrt{5\tan 72^\circ}$ (C) $40\sqrt{5\tan 72^\circ}$ (D) $20\sqrt{5\tan 36^\circ}$
(E) $40\sqrt{5\tan 36^\circ}$

---

35. Four women and six men from East Punxsutawney wish to enter IUP's 35th annual Co-ed Volleyball Tournament. This tournament has the unusual format of requiring teams to comprise five persons. The number of different ways that these ten people can form two teams with at least one woman on each team is:

(A) 60 (B) 120 (C) 150 (D) 180 (E) 720

---

36. The number of real solutions of the radical equation

$$2x - 5\sqrt{x} - 3 = 0$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

---

37. Consider the function f satisfying

$$f(2x-3) = x - 7.$$

The inverse of f is given by:

(A) $f^{-1}(x) = x - 7$ (B) $f^{-1}(x) = 2x - 3$ (C) $f^{-1}(x) = \dfrac{x-11}{2}$
(D) $f^{-1}(x) = 2x + 11$ (E) $f^{-1}(x) = 1$

---

38. There are two pumps drawing water from a tank. When the first pump works three hours and the second works five hours, 1350 cubic feet of water are withdrawn from the tank. When the first pump works four hours and the second works three hours, 1250 cubic feet of water are withdrawn. The number of cubic feet of water withdrawn in one hour by the second pump is:

(A) 150 (B) 170 (C) 190 (D) 200 (E) 210

---

39. The solution set of the equation

$$16^x + 32 = 9(2^{2x+1})$$

is:

(A) $\{\ \}$ (B) $\{-2/7\}$ (C) $\{(\log_2 9 - 4)/2\}$ (D) $\{1/2\}$
(E) none of these

---

40. The set of solutions of the radical equation

$$\sqrt{5x-4} - \sqrt{2x+1} = \sqrt{x-3}$$

is:

(A) $\{4\}$ (B) $\{1\}$ (C) $\{-1\}$ (D) $\{-1, 2\}$ (E) none of these

---

41. If the graph of $y = f(x)$ is translated to give the graph of $y - 3 = f(x+2)$, then the point $(-6, 2)$ is translated to the point:

(A) $(-8, 5)$ (B) $(-4, -1)$ (C) $(-8, -1)$ (D) $(-4, 5)$
(E) none of these

---

42. The first three numbers of an increasing arithmetic progression are added to the first three numbers of a geometric progression, respectively. The obtained sums are 85, 76, and 84, respectively. The sum of the first three terms of the arithmetic progression is 126. The first term of the geometric progression is:

(A) 17 (B) 34 (C) 51 (D) 75 (E) none of these

---

43. The number of distinct real solutions of the equation

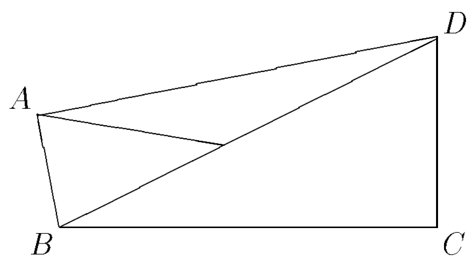
$$x^5 - 2x^4 + x^3 - 8x^2 + 16x - 8 = 0$$

is:

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

---

44. The quadrilateral $ABCD$ given in the figure has right angles at A and C. The lengths of three sides are $AB = 3$, $BC = 10$, and $CD = 5$. The length of the median from A in $\triangle ABD$ is:



(A) 5

(B) $\frac{5}{2}$

(C) $\frac{5\sqrt{5}}{2}$

(D) $\frac{15}{4}$

(E) none of these

---

45. The number of positive real numbers x that satisfy the equation

$$x + \log_x 2 = 1$$

is:

(A) 0 (B) 1 (C) 2 (D) infinite (E) none of these

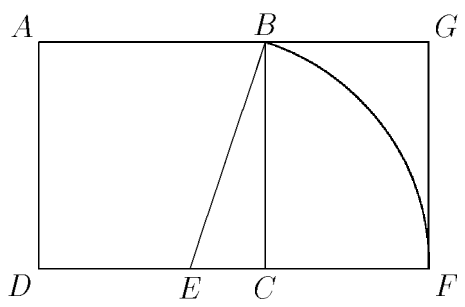
---

46. The front wheel of a wagon makes 110 more revolutions than the rear wheel in going a mile. If the circumference of the front wheel is increased by 25% and that of the rear wheel decreased 25%, the rear wheel then revolves 88 more times than the front wheel in going a mile. The original circumference of the front wheel in feet is:

(A) 8 (B) 10 (C) 12 (D) 14 (E) 16

---

47. In the diagram below, $ABCD$ is a square and BEF is a circular sector within the rectangle $AGFD$. If $AD = 3EC$, then the ratio of DF to AD is:



(A) 1

(B) $\frac{1+\sqrt{5}}{2}$

(C) $\frac{2+\sqrt{10}}{3}$

(D) $\frac{3+\sqrt{17}}{4}$

(E) 2

---

48. Dilbert spots a satellite in the sky. He knows its altitude is 100 miles. The angle of elevation (the angle from the horizontal to his line of sight) is about $6°$. Using the facts that the earth's radius is about 4000 miles and that $\sin 6° \approx .10$, his distance from the satellite, rounded to the nearest 100 miles, is:

(A) 400 miles (B) 600 miles (C) 800 miles (D) 1000 miles
(E) 1200 miles

---

49. If (x, y) is the ordered pair of real numbers that satisfies the system

$$\begin{aligned} \sqrt{x-y} &= x+y-7 \\ \sqrt{x+y} &= x-y-1, \end{aligned}$$

then the value of y is:

(A) 5/2 (B) 4 (C) 13/2 (D) 9 (E) 13/5

---

50. Let n be an integer greater than 1 and define

$$f(x) = \frac{1}{x-1} + \frac{2}{x-2} + \frac{3}{x-3} + \cdots + \frac{n}{x-n}.$$

The average of the zeros of f is:

(A) $\frac{n+1}{2}$ (B) $\frac{3n+2}{6}$ (C) $\frac{(3n+2)(n-1)}{6n}$ (D) $\frac{(3n+2)(n+1)}{6(n-1)}$
(E) none of these

Answer Key

1. E
2. B
3. B
4. C
5. A
6. D
7. D
8. C
9. A
10. A
11. A
12. D
13. D
14. C
15. B
16. D
17. B
18. B
19. B
20. E
21. E
22. D
23. A
24. D
25. D
26. E
27. A
28. C
29. D
30. E
31. D
32. C
33. B
34. E
35. B
36. B
37. D
38. A
39. E
40. A
41. A
42. E
43. B
44. C
45. A
46. C
47. C
48. B
49. A
50. B